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TWO PROGRAMS FOR THE HETEROGENEOUS TERMINAL DISTRIBUTION

by

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Introduction
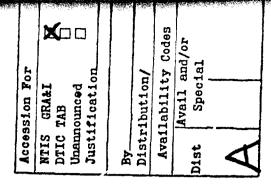
The following paper considers a generalisation of the heteregeneous battle model, introduced in Memoranda M7130 and M78106. The battle is heterogeneous in the sense that there may be more than one type of weapon unit on either side. The model allows the use of general attrition functions and control of battle end conditions. We show that the Taylor series method may be used to compute the probability distribution of the survivors as a function of time. We show also that the terminal distribution may be computed by an extension of the method used for the homogeneous case.

Specifications are given of two Fortran programs which compute the ultimate probabilities of battle outcome. These programs deal respectively with battles having two categories of weapon and three categories of weapon on each side. Either program allows 'degenerate' cases in which there are no weapons in given categories.

Force Levels and Valuation

Consider a battle between the Red force, having ρ categories of weapon unit, and the Blue force having σ categories of weapon unit. All units in any one category are identical. A unit is either a survivor, when it will be completely effective, or is a casualty, when it will be completely ineffective. Suppose that, initially, there are $M_1, M_2, \ldots, M_\rho$ units in the respective categories of the Red force, and $N_1, N_2, \ldots, N_\sigma$ units in the respective categories of the Blue force. At time t suppose that the numbers of survivors in the respective categories of the two forces are $\mu_1, \mu_2, \ldots, \mu_\rho$ and $\nu_1, \nu_2, \ldots, \nu_\sigma$. We have $0 \leq \mu_j \leq M_j$, for $1 \leq j \leq \rho$, and $0 \leq \nu_k \leq N_k$ for $1 \leq k \leq \sigma$.

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The 'vector' $(M_1, M_2, \ldots, M_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma)$ defines the Battle state at time t. Its two parts, the 'vectors' $(M_1, M_2, \ldots, M_\rho)$ and $(\nu_1, \nu_2, \ldots, \nu_\sigma)$, define the Red and Blue states at time t. When convenient, we shall write the vectors in the compact forms $(\bar{F}, \bar{\nu}), (\bar{F}), (\bar{\nu})$.

Suppose that the linear expression

$$a_1 M_1 + a_2 M_2 + \cdots + a_\rho M_\rho, \tag{1}$$

where the coefficients are not negative, assigns a 'value' to the Red state $(M_1, M_2, \ldots, M_\rho)$. We may choose the numbering of the weapon categories so that the coefficients a_j are decreasing. Moreover, there is no loss in generality in assuming the value of a weapon of the most valuable kind to be unity. We then have

$$1 = a_1 \geqslant a_2 \geqslant \cdots \geqslant a_\rho \geqslant 0. \tag{1a}$$

In a similar way suppose that the linear expression

$$b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\sigma \nu_\sigma, \tag{2}$$

where

$$1 = b_1 \geqslant b_2 \geqslant \cdots \geqslant b_\sigma \geqslant 0, \tag{2a}$$

assigns a 'value' to the Blue state.

We can now use value levels to classify the Battle states. The levels required are the Red loss level m and Red draw level m' and the Blue loss level n and Blue draw level n'. We must have

$$a_1 M_1 + a_2 M_2 + \cdots + a_\rho M_\rho > m' \geqslant m \neq 0, \tag{3}$$

and $\quad b_1 M_1 + b_2 M_2 + \cdots + b_\sigma M_\sigma > n' \geqslant n \geqslant 0.$

Battle ends either when the Red value falls to m or when the Blue value falls to n. The classification has the following precise form.

A state of 'battle continuation' is such that

$$a_1 M_1 + a_2 M_2 + \cdots + a_\rho M_\rho > m, \tag{4}$$

and $\quad b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\sigma \nu_\sigma > n.$

4

A state in which battle ends in Red victory or Red favoured draw is such that

$$a_1 \mu_1 + a_2 \mu_2 + \cdots + a_\rho \mu_\rho > m,$$

$$b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\sigma \nu_\sigma > n, \tag{5}$$

and there is a Blue weapon category λ, such that $1 + \nu_\lambda \le N_\lambda$, and

$$b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\lambda (1 + \nu_\lambda) + \cdots + b_\sigma \nu_\sigma \le n.$$

such a state is a Red victory when

$$a_1 \mu_1 + a_2 \mu_2 + \cdots + a_\rho \mu_\rho \ge m', \tag{6}$$

and is a Red favoured draw when

$$m' > a_1 \mu_1 + a_2 \mu_2 + \cdots + a_\rho \mu_\rho > m. \tag{7}$$

A state in which battle ends in Blue victory or Blue favoured draw is such that

$$b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\sigma \nu_\sigma > n,$$

$$a_1 \mu_1 + a_2 \mu_2 + \cdots + a_\rho \mu_\rho \le m, \tag{8}$$

and there is a Red weapon category ω such that $1 + \mu_\omega \le M_\omega$, and

$$a_1 \mu_1 + a_2 \mu_2 + \cdots + a_\omega (1 + \mu_\omega) + \cdots + a_\rho \mu_\rho > m.$$

Such a state is a Blue victory when

$$b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\sigma \nu_\sigma \ge n', \tag{9}$$

and is a Blue favoured draw when

$$n' > b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\sigma \nu_\sigma > n. \tag{10}$$

Note that, with the attrition laws we shall consider here, battle states which satisfy none of the above conditions occur only with probabality zero.

A homogeneous battle is one in which there is only one category of weapon on each side. The 'value' assigned to the Red or Blue state is then simply the number of Red or Blue survivors. For homogeneous battles the

classification of battle states may be displayed graphically as in the
diagram. Only integer points are relevant and, for consistency with
computer printed output, the diagram is 'graphwise' left-handed.

The Hannibal points' (after Hannibal c. 247 - 183 BC) are the points
of overwhelming victory. The 'Pyrrhic points' (after 'Pyrrhus c 318 - 272 BC)
are the points at which victory is barely attained. The Iwo Jima lines
(after the Battle of Iwo Jima, February 1945) are the lines of total
attrition.

If there are no reinforcements, then, during the course of the battle,
neither μ nor ν can increase. Hence, the battle state follows a track
which begins at the point (M, N) and moves either to the right or downwards.
Only tracks consisting of horizontal and vertical segments are significant.
Tracks containing oblique segments are not impossible but, with the
attrition laws considered here, such tracks occur only with probability
zero.

We may remark that the battle end conditions of the model are chosen

largely for computational simplicity. From other points of view, they may appear a little restrictive. In some situations, Hannibal points, at levels higher than the Pyrrhic points, are more realistic. Curves may replace the boundary lines joining the Hannibal and Pyrrhic points. However, such refinements add considerably to the computational task and the return for the extra effort is poor. Only slight changes are produced in the total probabilities of reaching the victory or draw boundaries.

The Attrition Law

Let the battle state be $(\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma)$ at time t. If this state is a state of continuation, suppose that in the short interval of time h casualties occur with the following probabilities. A single casualty in category k of the Blue force, but no other casualty to either force, occurs with probability

$$h \phi_k(\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma) + O(h^2), \quad k = 1, 2, \ldots, \sigma, \quad (11)$$

where $\phi_k(\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma)$ is a given function of $\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma$. A single casualty in category j of the Red force, but no other casualty to either force, occurs with probability

$$h \psi_j(\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma) + O(h^2), \quad j = 1, 2, \ldots, \rho, \quad (12)$$

where $\psi_j(\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma)$ is a given function of $\kappa_1, \kappa_2, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma$. A multiple casualty of any type occurs with probability of order $O(h^2)$.

The functions $\phi_k, k = 1, 2, \ldots, \sigma$, and $\psi_j, j = 1, 2, \ldots, \rho$, are known as the attrition functions. The ϕ_k determine the capability of the Red force and the ψ_j determine the capability of the Blue force. These functions are subject to the following restrictions. Neither the ϕ_k nor the ψ_j can be negative. If all κ_j are zero, then all ϕ_k are zero. If all ν_k are zero, then all ψ_j are zero. If, for given j, $\kappa_j = 0$,

7

then $V_j = 0$, and if, for given k, $\nu_k = 0$, then $\phi_k = 0$.

The functions ϕ_k will, in general, tend to increase with increase of the μ. These functions are more dependent on the μ than they are on the ν and, with the exception that ϕ_k vanishes when ν_k vanishes, they may be completely independent of the ν. Similar remarks, with the roles of μ and ν interchanged, apply to the functions ψ_j.

Differential Equations for the Probabilities of Battle States

Let $P(\mu_1, \kappa_1, \ldots, \kappa_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma, t) \equiv P(\bar{\mu}, \bar{\nu}, t)$ be the probability that the battle state is $(\bar{\mu}, \bar{\nu})$ at time t. Note that $P(\bar{\mu}, \bar{\nu}, t) = 0$ when for some j, $\kappa_j > M_j$, or for some k, $\nu_k > N_k$.

If the state $(\bar{\mu}, \bar{\nu})$ is one of continuation we have
$$P(\bar{\mu}, \bar{\nu}, t+h) = h \sum_{k=1}^{\sigma} \phi_k(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma) P(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma, t)$$
$$+ h \sum_{j=1}^{\rho} \psi_j(\kappa_1, \kappa_2, \ldots, 1+\kappa_j, \ldots, \kappa_\rho, \bar{\nu}) P(\kappa_1, \kappa_2, \ldots, 1+\kappa_j, \ldots, \kappa_\rho, \bar{\nu}, t)$$
$$+ P(\bar{\mu}, \bar{\nu}, t) \prod_{k=1}^{\sigma}(1 - h\phi_k(\bar{\mu}, \bar{\nu})) \prod_{j=1}^{\rho}(1 - h\psi_j(\bar{\mu}, \bar{\nu})) + O(h^2). \qquad (13)$$

The first term on the right of equation (13) is the probability that, in the short time interval h the battle state $(\bar{\mu}, \bar{\nu})$ arises by a single casualty to the Blue force. The second term is the probability that the state arises by a single casualty to the Red force. The third term is the probability that the state $(\bar{\mu}, \bar{\nu})$ exists at time t and does not change in the time interval h. Note that the conditions $\phi_k = 0$ when $\nu_k = 0$, and $\psi_j = 0$ when $\mu_j = 0$, are required in this term. All other modes which give rise to the state $(\bar{\mu}, \bar{\nu})$ have probability of order $O(h^2)$.

When equation (13) is re-arranged and h tends to zero we obtain
$$P'(\bar{\mu}, \bar{\nu}, t) + \left\{ \sum_{k=1}^{\sigma} \phi_k(\bar{\mu}, \bar{\nu}) + \sum_{j=1}^{\rho} \psi_j(\bar{\mu}, \bar{\nu}) \right\} P(\bar{\mu}, \bar{\nu}, t)$$
$$= \sum_{k=1}^{\sigma} \phi_k(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma) P(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma, t)$$
$$+ \sum_{j=1}^{\rho} \psi_j(\kappa_1, \kappa_2, \ldots, 1+\kappa_j, \ldots, \kappa_\rho, \bar{\nu}) P(\kappa_1, \kappa_2, \ldots, 1+\kappa_j, \ldots, \kappa_\rho, \bar{\nu}, t). \qquad (14)$$

8

Equation (14) requires modification when the battle state $(\bar{F}, \bar{\nu})$ is one of end in Red victory or Red favoured draw. If a casualty to the Blue force gives rise to the state, then that casualty occurs in weapon category λ such that $1 + \nu_\lambda \leqslant M_\lambda$ and $b_1 \nu_1 + b_2 \nu_2 + \cdots + b_\lambda (1 + \nu_\lambda) + \cdots + b_r \nu_r > n$. There is a greatest value for λ with this property. Battle may end by a Blue casualty in any weapon category $k \leqslant \lambda$ such that $\nu_k \neq M_k$. Battle cannot end by a Red casualty. If λ has its greatest possible value, we now have

$$P(\bar{F}, \bar{\nu}, t+h) = h \sum_{k=1}^{\lambda} b_k (\bar{F}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_r) P(\bar{F}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_r, t)$$

$$+ P(\bar{F}, \bar{\nu}, t) + O(h^2). \tag{15}$$

The first term on the right of equation (15) is the probability that, in the time interval h, the battle state (F, ν) arises by a single casualty to the Blue force. Note that any term under the summation sign having $1 + \nu_k > M_k$ is zero. The second term on the right of equation (15) is the probability that state $(\bar{F}, \bar{\nu})$ exists at time t. The state cannot change in the time interval h. From equation (15) we have

$$P'(\bar{F}, \bar{\nu}, t) = \sum_{k=1}^{\lambda} b_k (\bar{F}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_r) P(\bar{F}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_r, t). \tag{16}$$

If the battle state $(\bar{F}, \bar{\nu})$ is one of end in Blue victory or Blue favoured draw, then, by a similar argument, we have

$$P'(\bar{F}, \bar{\nu}, t) = \sum_{j=1}^{\omega} \psi_j (\mu_1, \mu_2, \ldots, 1+\mu_j, \ldots, \mu_\rho, \bar{\nu}) P(\mu_1, \mu_2, \ldots, 1+\mu_j, \ldots, \mu_\rho, \bar{\nu}, t), \tag{17}$$

where ω is the largest integer such that $1 + \mu_\omega \leqslant M_\omega$ and

$$a_1 \mu_1 + a_2 \mu_2 + \cdots + a_\omega (1 + \mu_\omega) + \cdots + a_\rho \mu_\rho > m.$$

The probability distribution $P(\bar{F}, \bar{\nu}, t)$ is determined completely by the equations (14), (16), (17), together with the boundary conditions

9

$$P(\mu_1, \kappa_1, \ldots, 1+m_j, \ldots, \mu_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma, t) = 0, \quad j = 1, 2, \ldots, \rho,$$

$$P(\mu_1, \kappa_1, \ldots, \mu_\rho, \nu_1, \nu_2, \ldots, 1+n_k, \ldots, \nu_\sigma, t) = 0, \quad k = 1, 2, \ldots, \sigma,$$

(18)

and the initial conditions

$$P(m_1, m_2, \ldots, m_\rho, N_1, N_2, \ldots, N_\sigma, 0) = 1,$$

(19)

and

$$P(\mu_1, \kappa_1, \ldots, \mu_\rho, \nu_1, \nu_2, \ldots, \nu_\sigma, 0) = 0,$$

where $(\bar{\mu}, \bar{\nu}) \neq (\bar{n}, \bar{N})$.

Equations (14), (16), (17) taken over the battle states form a linear system of differential equations with constant coefficients. In principle, the equations can be solved by elementary methods, but this is practical only in cases with small numbers of battle states. However, the qualitative nature of the solution can be obtained without solving the equations in detail.

Let $\mu_1 + \kappa_1 + \cdots + \mu_\rho + \nu_1 + \nu_2 + \cdots + \nu_\sigma$ define the order of the battle state $(\bar{\mu}, \bar{\nu})$ and consider first the states of battle continuation. The two terms on the left-hand side of equation (14) have the order of the state to which the equation refers. All the terms on the right-hand side have order exceeding this by unity. Hence, equation (14) is a linear differential equation of the first order in $P(\bar{\mu}, \bar{\nu}, t)$ and this equation may be solved for $P(\bar{\mu}, \bar{\nu}, t)$ when the P of the higher order are known.

The boundary conditions (18) have the effect of removing any terms of order exceeding $\mu_1 + \kappa_1 + \cdots + \mu_\rho + N_1 + N_2 + \ldots + N_\sigma$ from equation (14). In particular, the equation for $P(m_1, m_2, \ldots, m_\rho, N_1, N_2, \ldots, N_\sigma)$ is very simple and has a single exponential term, with negative exponent, as its solution. An inductive argument now shows that, for any state of battle continuation,

$P(\vec{\mu}, \vec{\nu}, t)$ is a sum of exponential terms having negative exponents. Except for some unrealistic attrition functions, the coefficients of the terms are constants. These functions will be excluded by a condition that the coefficient of $P(\vec{\mu}, \vec{\nu}, t)$ in equation (14) shall increase with increase of any μ and ν. The initial conditions (19) imply that, for $(\vec{\mu}, \vec{\nu}) \neq (\vec{m}, \vec{n})$ the sum of the constant coefficients is zero. Note that as time becomes infinite all terms tend to zero.

A similar argument applies to the states of battle end. However, there is now only the derivative on the left-hand side of equations (16) and (17). Hence, P consists of a constant of integration together with a sum of exponential terms with negative exponents. When time tends to infinity, P tends to the constant of integration. The constant is not negative since it is a probability. Further, since equations (16) and (17) imply that the derivative P' is positive, P increases steadily to the constant.

Note that the constants of integration form a probability distribution over the states of battle end. This distribution is known as the 'terminal distribution'. It is the limiting form of the distribution $P(\vec{\mu}, \vec{\nu}, t)$ as t becomes infinite, and is, perhaps, the most important special case.

Computation of $P(\vec{\mu}, \vec{\nu}, t)$

The differential equations (14), (16), (17) can be solved numerically by a step by step integration process. The Taylor series method is particularly convenient. We have

$$P(\vec{\mu}, \vec{\nu}, t+h) = P(\vec{\mu}, \vec{\nu}, t) + \frac{h}{1!} P'(\vec{\mu}, \vec{\nu}, t) + \frac{h^2}{2!} P''(\vec{\mu}, \vec{\nu}, t) + \cdots$$

$$= U^0(\vec{\mu}, \vec{\nu}, t) + U'(\vec{\mu}, \vec{\nu}, t) + U^2(\vec{\mu}, \vec{\nu}, t) + \cdots \qquad (20)$$

where $U^r(\vec{\mu}, \vec{\nu}, t) = \frac{h^r}{r!} P^r(\vec{\mu}, \vec{\nu}, t)$, $r = 1, 2, \cdots$.

11

There are no singularities in the differential equations so that the series (20) converges for all t .

Suppose that $(\bar{\mu}, \bar{\nu})$ is a state of battle continuation. Then, by forming the derivative of order $r-1$ of equation (14), multiplication by $\frac{t}{r!}$ and rearrangement, we obtain

$$U^r(\bar{\mu},\bar{\nu},t) = \frac{t}{r}\left\{\sum_{\lambda=1}^{r}\Delta\nu_\lambda\left(\phi_\lambda(\bar{\mu},\bar{\nu})U^{r-1}(\bar{\mu},\bar{\nu},t)\right) + \sum_{j=1}^{\rho}\Delta\mu_j\left(\psi_j(\bar{\mu},\bar{\nu})U^{r-1}(\bar{\mu},\bar{\nu},t)\right)\right\}, \quad (21)$$

where $\Delta\mu_\lambda$ and $\Delta\nu_j$ are forward partial difference operators with respect to μ_λ and ν_j respectively. The boundary conditions (18) ensure that the equation holds when $\mu_j = M_j$ or $\nu_\lambda = N_\lambda$.

Suppose that $(\bar{\mu}, \bar{\nu})$ is a state of battle end in Red's favour. Then in a similar manner we obtain from equation (16) that

$$U^r(\bar{\mu},\bar{\nu},t) = \frac{t}{r}\sum_{\lambda=1}^{\lambda}\phi_\lambda(\bar{\mu},\nu_1,\nu_2,\ldots,1+\nu_\lambda,\ldots,\nu_\sigma)U^{r-1}(\bar{\mu},\nu_1,\nu_2,\ldots,1+\nu_\lambda,\ldots,\nu_\sigma,t). \quad (22)$$

Similarly, if $(\bar{\mu}, \bar{\nu})$ is a state of battle end in Blue's favour, then from equation (17)

$$U^r(\bar{\mu},\bar{\nu},t) = \frac{t}{r}\sum_{j=1}^{\omega}\psi_j(\mu_1,\mu_2,\ldots,1+\mu_j,\ldots,\mu_\rho,\bar{\nu})U^{r-1}(\mu_1,\mu_2,\ldots,1+\mu_j,\ldots,\mu_\rho,\bar{\nu},t). \quad (23)$$

If for a given value of t the values of the $P(\bar{\mu},\bar{\nu},t)$ are known, equations (21), (22) may be used recursively to compute the terms $U^r(\bar{\mu},\bar{\nu},t)$ of series(20). Hence $P(\bar{\mu},\bar{\nu},t+t)$ may be determined and, by taking sufficient terms, the truncation error may be reduced to any required level. Beginning with the known initial values $P(\bar{\mu},\bar{\nu},0)$ the process provides a step by step method of computing $P(\bar{\mu},\bar{\nu},t)$ at any required interval of tabulation in t.

The method requires only two locations of computer storage per battle state. A large step of integration may be used. However there is an optimum

step for which the arithmetic required and the effects of rounding errors are minimum. The general solution of equations (14), (16), (17) does not contain exponential terms with positive exponents. Hence, unwanted components of the solution, introduced by the rounding errors, do not tend to build up as integration proceeds.

Computation of the Terminal Distribution

When t tends to infinity, the distribution $P(\bar{\mu}, \bar{\nu}, t)$ tends to zero for all states of battle continuation. The limiting values are non-zero over the states of battle end and form the 'terminal distribution'. This is the distribution of battle outcome and can be computed independently of the differential equations (14), (16), (17).

Let $w(\bar{\mu}, \bar{\nu})$ be the ultimate probability of a battle end in state $(\bar{\mu}, \bar{\nu})$ in favour of the Red force. The state $(\bar{\mu}, \bar{\nu})$ satisfies conditions (5). Let $v(\bar{\mu}, \bar{\nu})$ be the ultimate probability of a battle end in state $(\bar{\mu}, \bar{\nu})$ in favour of the Blue force. The state $(\bar{\mu}, \bar{\nu})$ satisfies conditions (8).

Let
$$\bar{P}(\bar{\mu}, \bar{\nu}, s) = \int_0^\infty e^{-st} P(\bar{\mu}, \bar{\nu}, t)\, dt$$
be the Laplace transform of $P(\bar{\mu}, \bar{\nu}, t)$. We have that the Laplace transform of the derivative $P'(\bar{\mu}, \bar{\nu}, t)$ is $s\bar{P}(\bar{\mu}, \bar{\nu}, s) - P(\bar{\mu}, \bar{\nu}, 0)$, the Laplace transform of unity is $\frac{1}{s}$ and the Laplace transform of $e^{-\alpha t}$ is $\frac{1}{s+\alpha}$.

Suppose that $(\bar{\mu}, \bar{\nu})$ is a state of battle end in favour of the Red force. Then $P(\bar{\mu}, \bar{\nu}, t)$ is equal to $w(\bar{\mu}, \bar{\nu})$ together with a sum of terms of the form $c\, e^{-\alpha t}$. Hence the Laplace transform $\bar{P}(\bar{\mu}, \bar{\nu}, s)$ will consist of $\frac{w(\bar{\mu}, \bar{\nu})}{s}$ together with a sum of terms of the form $\frac{c}{s+\alpha}$. It follows that
$$w(\bar{\mu}, \bar{\nu}) = \lim_{s \to 0} s \bar{P}(\bar{\mu}, \bar{\nu}, s).$$

13

The Laplace transform of equation (16), and the initial condition $P(\bar{\mu}, \bar{\nu}, 0) = 0$, gives

$$s\,\bar{P}(\bar{\mu}, \bar{\nu}, s) = \sum_{k=1}^{\lambda} \phi_k(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma)\,\bar{P}(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma, s). \quad (24)$$

Hence, when s tends to zero we have

$$w(\bar{\mu}, \bar{\nu}) = \sum_{k=1}^{\lambda} \phi_k(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma)\,u(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma), \quad (25)$$

where $u(\bar{\mu}, \bar{\nu}) = \bar{P}(\bar{\mu}, \bar{\nu}, 0)$, for states of battle continuation.

If $(\bar{\mu}, \bar{\nu})$ is a state of battle end in favour of the Blue force, then equation (17) implies, by a similar argument, that

$$v(\bar{\mu}, \bar{\nu}) = \sum_{j=1}^{\omega} \psi_j(\mu_1, \mu_2, \ldots, 1+\mu_j, \ldots, \mu_\rho, \bar{\nu})\,u(\mu_1, \mu_2, \ldots, 1+\mu_j, \ldots, \mu_\rho, \bar{\nu}). \quad (26)$$

Note that, in equations (25) and (26), the boundary conditions (18) imply that $u(\bar{\mu}, \bar{\nu}) = 0$ when any μ_j exceeds M_j, or any ν_k exceeds N_k.

Suppose that $(\bar{\mu}, \bar{\nu})$ is a state of battle continuation with at least one casualty. The Laplace transform of equation (14) and the initial conditions (19) imply that, when s becomes zero,

$$u(\bar{\mu}, \bar{\nu})\left(\sum_{k=1}^{\sigma} \phi_k(\bar{\mu}, \bar{\nu}) + \sum_{j=1}^{\rho} \psi_j(\bar{\mu}, \bar{\nu}) \right)$$

$$= \sum_{k=1}^{\sigma} \phi_k(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma)\,u(\bar{\mu}, \nu_1, \nu_2, \ldots, 1+\nu_k, \ldots, \nu_\sigma)$$

$$+ \sum_{j=1}^{\rho} \psi_j(\mu_1, \mu_2, \ldots, 1+\mu_j, \ldots, \mu_\rho, \bar{\nu})\,u(\mu_1, \mu_2, \ldots, 1+\mu_j, \ldots, \mu_\rho, \bar{\nu}). \quad (27)$$

As in the case of equations (25) and (26), we have $u(\bar{\mu}, \bar{\nu}) = 0$ when any μ_j exceeds M_j, or any ν_k exceeds N_k.

14

When $(\bar{F},\bar{D})=(\bar{H},\bar{N})$ equation (14) reduces to

$$P'(\bar{H},\bar{N},t)+\left\{\sum_{h=1}^{\sigma}\phi_h(\bar{H},\bar{N})+\sum_{j=1}^{\rho}\psi_j(\bar{H},\bar{N})\right\}P(\bar{H},\bar{N},t)=0.$$

The Laplace transform of this equation and the initial conditions (19)
lead to

$$u(\bar{H},\bar{N})=\frac{1}{\sum_{h=1}^{\sigma}\phi_h(\bar{H},\bar{N})+\sum_{j=1}^{\rho}\psi_j(\bar{H},\bar{N})}\tag{28}$$

Equation (27) provides a recursive relation and equation (28) provides

the initial value from which $u(\bar{H},\bar{D})$ can be computed for any state of

battle continuation. Equations (25) and (26) can then be used to compute

the two parts of the terminal distribution $w(\bar{H},\bar{D})$ and $v(\bar{H},\bar{D})$.

The process is accurate since quantities arising during computation

are positive and no differences between nearly equal large quantities are

formed. Computer storage requirements are much smaller than those of the

general method.

Remarks on 'Valuation' and Attrition Functions

In applications, the valuation coefficients a_j and b_h and the loss and

draw levels must fit the circumstance and the questions under analysis.

In the first instance, the user should decide qualitatively for each force

the relative worth of the various weapon categories. The coefficients can

then be assigned to the qualitative ordering. Valuation is used to assign

conditions for victory, loss or draw. It is not necessarily a measure of

effectiveness.

A loss level of about two-thirds of the initial valuation and a draw

level of about three-quarters of the initial valuation are reasonable

assumptions in a wide range of cases. There are seldom good reasons for

using loss levels close to total attrition. However, there are cases when

the Red and Blue loss levels are different. For example, the roles of

attacker and defender must be considered.

The choice of the attrition functions ϕ_λ and γ_j may be difficult. In the first instance linear expressions in the μ alone should be used for the ϕ_λ , but these expressions must be replaced by zero when they become negative or when $\gamma_\lambda = 0$. Different expressions may be used for different parts of the working range. Similar remarks, with the roles of μ and ν interchanged, apply to the γ_j . An assessment of the relative effectiveness of the weapons in the various categories must be made and the constants in the linear expressions for ϕ_λ and γ_j chosen to match this assessment. Note that the terminal distribution is not alerted when all the functions ϕ_λ and γ_j are multiplied by the same constant.

The Three by Three Battle

The method of computing the terminal distribution has been incorporated into a Fortran computer program, RAO228, which deals with a battle having three categories of weapon on each side. The user may choose the attrition functions at will, but must prepare suitable program segments to evaluate them.

The terminal distribution over the states of battle end is generally so extensive that printing is impractical. Instead, when the end is in Red's favour, the program prints a condensed distribution over the valuation of the Red states. Similarly,when the end is in Blue's favour, the program prints a condensed distribution over the valuation of the Blue states. The probabilities of Red victory Red favoured draw, Blue victory, and Blue favoured draw are printed separately.

The user must provide Fortran function segments to calculate the attrition functions $\phi_1, \phi_\lambda, \phi_j, \gamma_1, \gamma_\lambda, \gamma_j$. A subroutine to input any numerical data required by these segments, and a subroutine to print

16

identification of the functions are also required. Communications between these segments should be via a common block.

The function segment $PHI(MVI, MV2, MV3, NVI, NV2, NV3)$ must be provided to compute the attrition function $\phi_1(M_1, M_2, M_3, \nu_1, \nu_2, \nu_3)$. The segment must ensure that $\phi_1 = 0$, when $M_1 = M_2 = M_3 = 0$, or when $\nu_1 = 0$. Similarly, function segments $PH2$ and $PH3$ must be provided to compute ϕ_2 and ϕ_3 and function segments $PSI, PS2, PS3$ must be provided to compute ψ_1, ψ_2, ψ_3.

When the segments $PHI, PH2, \ldots, PS3$ require numerical data which varies from case to case, the user must provide a subroutine REDAT to input this data. The subroutine should read the data from cards via channel 1 and place it in a common block. When no such data is required, the subroutine should consist of the single instruction RETURN.

Definitions of the attrition functions and values of any associated parameters should be printed as part of the program output. For this purpose the user must provide subroutine PRDAT which sends output to the line printer via channel 2. If no information about the attrition functions is to be printed the subroutine should consist of the single instruction RETURN.

The program will handle cases in succession. For each case the main program itself reads seven cards bearing the following information.

Card 1   Case Identifier in columns

Card 2   Initial Red levels:- M_1 in columns 1-10, M_2 in cols 11-20, M_3 in cols 21-30

Card 3   Red valuation coefficients :-

        A_1 in cols 1-10 (dec point col 7)

        A_2 "   " 11-20 ("   "   " 17)

        A_3 " " 21-30 ("   "   " 27)

Card 4   Red Loss and Draw ratios and printing interval :-

        Loss ratio in cols 1-10 (dec point col 7)

        Draw ratio " " 11-20 ("   "   " 17)

        Printing increment in cols 21-30 (dec point col 27)

Cards 5, 6, 7 are similar respectively to cards 2, 3, 4 but give the appropriate parameter values for Blue. The cards holding the information to be input by the subroutine REDAT follow and complete the data required for the case.

Cards for other cases may follow. The program stops when a card with -1 in columns 2 & 3 is read.

Output from the program is in two sections. The first of these gives the parameters input by the main program, the information about the attrition functions provided by subroutine PRDAT, and the computed probabilities of Red victory, Blue victory, and draw.

The second section gives a condensation of the terminal distribution over the states of battle end. In the event of battle end in favour of Red the distribution of valuation of the survivors is tabulated in cumulative form. A similar tabulation is given for battle end in favour of Blue.

The Two by Two Battle

Computer program RA0229 uses the same general method for a battle having two categories of weapon on each side as does program RA0228 for the three by three case. There are some differences in the printing since, in this case it is practical to print the whole of the terminal distribution. The program will also print force names and weapon types. The most 'valuable' weapons are assumed to be in category 1 and have unit value.

The user must provide Fortran function segments to calculate the attrition functions ϕ_1, ϕ_2, ψ_1, ψ_2 . Subroutines to input any numerical data required by these segments and to print identification of the functions are also required. These segments should communicate via a common block.

The function segment $PHI(MU1, MU2, NU1, NU2)$ must be provided to compute the attrition function $\phi_1(\mu_1, \mu_2, \nu_1, \nu_2)$. The segment must ensure that $\phi_1 = 0$ when $\mu_1 = \mu_2 = 0$, or when $\nu_1 = 0$. Similarly, function segments $PH2, PS1, PS2$ must be provided to compute the attrition functions ϕ_2, ψ_1, ψ_2 respectively.

When the segments $PHI, PH2, PS1, PS2$ require numerical data, the user must provide a subroutine REDAT to input this data. The subroutine should read the data from cards via channel 1 and place it in a common block. When no such data is required, the subroutine should consist of the single instruction RETURN.

Definitions of the attrition functions and values of parameters should be printed with the program output by subroutine PRDAT. This subroutine should send the output to the lineprinter via channel 2. If no information

19

regarding the attrition functions is to be printed the subroutine should consist of the single instruction RETURN.

The program will handle cases in succession and provision has been made to assign force names and weapon names to a series of cases. At the beginning of a series the program reads the following four cards.

Card 1   has the indicator 1 in column 3.

Card 2   has the name of the Red force right justified in columns 1-24 and the name of the Blue force right justified in columns 25-48.

Card 3   has the names of the first and second categories of Red weapons left justified in columns 1-40, and columns 41-80, respectively.

Card 4   has the names of the first and second categories of Blue weapons left justified in columns 1-40, and columns 41-80, respectively.

The names set by these cards remain set until another indicator card is read.

For each case the program reads the following five cards

Card 1   Case Identifier in columns 4-75

Card 2   Initial Red levels

M1 in columns 1-10, M2 in columns 11-20

Card 3   Valuation coefficient of second Red weapon category in columns 1-10 (decimal point column 7)

Red's loss valuation ratio in columns 11-20 (decimal point in column 17)

Red's draw valuation ratio in columns 21-30 (decimal point in column 27)

Red's valuation printing interval in columns 31-40

(decimal point in column 37)

Cards 4 and 5 are similar to cards 2 and 3 but give the values of the Blue parameters.

Cards holding the information to be read by subroutine REDAT follow. This completes the data for one case.

Cards for other cases may follow. The program stops when a card with -1 in columns 2 and 3 is read.

Output from the program is in four sections. The first of those gives the parameters input by the main program, the information about the attrition functions provided by subroutine PRDAT, and the computed probabilities of Red victory, Blue victory, and draw.

The second section gives, for battle end in Red's favour, the bivariate probability distribution over the Red survivors. The third section gives, for battle end in Blue's favour, the bivariate probability distribution over the Blue survivors. These sections together give the complete terminal distribution.

The fourth section gives a condensation of the terminal distribution as for program RAO228. For battle end in Red's favour the distribution of valuation of the survivors is tabulated in cumulative form. A similar tabulation is given for battle end in favour of Blue.